MATHEMATICS

# ISOLATED CRITICAL POINTS OF $C^{\infty}$ AND $C^{\omega}$ FUNCTIONS


BY

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## § 1. *Introduction.*

In this paper we give examples of isolated critical points of $C^{\infty}$ functions, which are essentially different from any isolated critical point of a $C^{\omega}$ (=analytic) function. I.e. we define an equivalence relation between isolated critical points and show that there are equivalence classes in which there is no isolated critical point of a $C^{\omega}$ function.

Definition 1. Let $f$ and $g$ be $C^{\infty}$ functions defined on $U_f$ and $U_g$, open subspaces in $\mathbf{R}^n$, and let $p \in U_f$ be an isolated critical point of $f$ and $q \in U_g$ an isolated critical point of $g$. $(f, p)$ and $(g, q)$ are called *topologically equivalent* if there exists a homeomorphism $\varphi$ of a neighbourhood $U(p)$ of $p$ in $U_f$ onto a neighbourhood $U(q)$ of $q$ in $U_g$ such that $\varphi(U(p) \cap f^{-1}(-\infty, f(p)]) = U(q) \cap g^{-1}(-\infty, g(q)]$.

In [2] it is proved, that for any $C^{\omega}$ function $f$ and any $a \in \mathbf{R}$ the inverse image $f^{-1}(a)$ can be triangulated. Consequently there exists for every isolated critical point $p$ of a $C^{\omega}$ function $f : U \to \mathbf{R}$, a closed neighbourhood $U(p)$ of $p$ in $f^{-1}(f(p))$ which is homeomorphic with the cone on its boundary. As $\varphi$ in definition 1 induces a local homeomorphism of $f^{-1}(f(p))$ on $g^{-1}(g(q))$, the critical level of every isolated critical point, which is topologically equivalent with some isolated critical point of a $C^{\omega}$ function, must be tame in the critical point.

Definition 2. Let $V$ be a topological space. $V$ is called *tame* in a point $p \in V$ if $p$ has a closed neighbourhood which is homeomorphic with the cone on its boundary.

In this paper we give a proof of the following

Theorem: For every $n \geqslant 4$ there exists a $C^{\infty}$ function $f : \mathbf{R}^n \to \mathbf{R}$ with an isolated critical point $p$ whose level is not tame at $p$.

Corollary: There are isolated critical points of $C^{\infty}$ functions on $\mathbf{R}^n$ for every $n \geqslant 4$, which are not topologically equivalent with any isolated critical point of a $C^{\omega}$ function.

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§ 2. *The analytic part of the proof.*

In this § we reduce the problem to a purely topological one:

Proposition 1: If, for some $n$, there exists a triple $Q \subset \mathbf{R}^{n-1} \subset \mathbf{R}^n$ ($\mathbf{R}^{n-1} = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | x_n = 0\} \subset \mathbf{R}^n$) for which

(i) there exists a continuous map $\tau : \mathbf{R}^n \to \mathbf{R}^n$ with $\tau(Q) = \mathbf{0} = (0, \ldots, 0)$ and the restriction $\tau | \mathbf{R}^n - Q$ is a diffeomorphism onto $\mathbf{R}^n - \{\mathbf{0}\}$,

(ii) there is no closed neighbourhood $U$ of $Q$ in $\mathbf{R}^{n-1}$ such that $U - Q$ is homeomorphic with $\partial U \times (0, 1]$,

then there exists a $C^\infty$ function $f : \mathbf{R}^n \to \mathbf{R}$ with an isolated critical point in $\mathbf{0}$ whose level is not tame in $\mathbf{0}$.

Proof: We shall use the map $\tau$ to construct a $C^\infty$ map $f : \mathbf{R}^n \to \mathbf{R}$ with only one critical point (in $\mathbf{0}$) and with $f^{-1}(0) = \tau(\mathbf{R}^{n-1})$. If $f^{-1}(0)$ is tame in $\mathbf{0}$, then $\mathbf{0}$ has a closed neighbourhood $W$ in $f^{-1}(0) = \tau(\mathbf{R}^{n-1})$ homeomorphic with a cone on its boundary, and $Q$ has a closed neighbourhood $\tau^{-1}(W)$ in $\mathbf{R}^{n-1}$ with $\tau^{-1}(W) - Q$ homeomorphic with $\partial(\tau^{-1}(W)) \times (0, 1]$. This contradicts assumption (ii). Hence $f^{-1}(0)$ is not tame in $\mathbf{0}$.

In order to obtain the function $f$ we first construct a suitable $C^\infty$ function $F : \mathbf{R}^n \to \mathbf{R}$ with $F^{-1}(0) = \mathbf{R}^{n-1}$ and $(\partial/\partial x_n)(F(x)) = 0$ iff $x \in Q$; we then define $f$ by $F = \tau f$.

As $F$ is constant on $Q$ the function $f$ is indeed well defined. Moreover $f^{-1}(0)$ is not tame at $\mathbf{0}$. If $f$ turns out to be $C^\infty$, it is the required function.

For the construction of $F$ we use a basis $\{Q_i\}_{i=1}^\infty$ of compact neighbourhoods of $Q$ in $\mathbf{R}^{n-1}$ with $Q_i \subset \operatorname{int} Q_{i-1}$;
a partition of unity $\{\lambda_i\}_{i=1}^\infty$ of $\mathbf{R}^{n-1} - Q$ with support $\lambda_1 \subset$ complement of $Q_2$, support $\lambda_i \subset Q_{i-1} - \operatorname{int} Q_{i+1}$ for all $i > 1$;
and a sequence $\{h_i\}_{i=1}^\infty$ of functions on $\mathbf{R}^1$ with

$$h_i(x) = x \text{ if } |x| \geqslant (i)^{-1}$$

$$h_i(x) = 0 \text{ if } |x| \leqslant (i+1)^{-1}$$

$$(d/dx)(h_i(x)) \geqslant 0 \text{ for all } x.$$

$F$ will be of the form:

$$F(x_1, \ldots, x_n) = \left[ \sum_{i=1}^{\infty} (a_i \cdot \lambda_i(x_i, \ldots, x_{n-1})) \right] \cdot x_n + \sum_{j=1}^{\infty} b_j \cdot h_j(x_n),$$

where all $a_i$ and $b_j$ are positive real numbers.

We see that $F^{-1}(0) = \mathbf{R}^{n-1}$. Note that $F$ is not $C^\infty$ for all sequences $\{a_i\}$ and $\{b_j\}$. However if $F$ is a $C^\infty$ function then $\dfrac{\partial F(x)}{\partial x_n} = 0$ implies $x \in Q$.

We now compute conditions for $\{a_i\}$ and $\{b_j\}$ to make $F$ and $f$ $C^\infty$. As $\mathbf{0} \notin$ support $h_j$ and $Q \cap$ support $\lambda_i = \emptyset$ one can define $C^\infty$ functions $\Lambda_i$

and $H_j$ on $\mathbf{R}^n$ with $\mathbf{0} \notin \text{support } \Lambda_i$ and $\mathbf{0} \notin \text{support } H_j$, by

$$\Lambda_i(\tau(x_1, \ldots, x_n)) = \lambda_i(x_1, \ldots, x_{n-1}) \cdot x_n,$$
$$H_j(\tau(x_1, \ldots, x_n)) = h_j(x_n), \text{ for all } i, j \geqslant 1.$$

We define $D_k(\varphi)$ for any $C^\infty$ function $\varphi : \mathbf{R}^n \to \mathbf{R}$ with $\mathbf{0} \notin \text{support } \varphi$ by

$$D_k(\varphi) = \max \left\{ \left| \left( \sum_{j=1}^{n} x_j^2 \right)^{-1} \left( \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_p}} \varphi(x_1, \ldots, x_n) \right) \right| \right\},$$

where the maximum is taken over the set

$$\{i_1, \ldots, i_p, x_1, \ldots, x_n \mid p \leqslant k, \sum_{j=1}^{n} x_j^2 \leqslant 1\}.$$

If we choose

$$0 < a_i < [\max(1, D_i(\Lambda_i))]^{-1}$$

and

$$0 < b_j < [\max(1, 2^j D_j(H_j))]^{-1},$$

$F$ and $f$ will be $C^\infty$ functions.

## § 3. *The construction of* $Q \subset \mathbf{R}^3 \subset \mathbf{R}^4$.

$Q \subset \mathbf{R}^3$ will be example 1, 1 in [1]; i.e. $Q$ is a wild arc in $\mathbf{R}^3$ with $\pi_1(\mathbf{R}^3 - Q) \neq 0$ and $H_1(\mathbf{R}^3 - Q) = 0$. By the properties of $Q$ there cannot exist a closed neighbourhood $U$ of $Q$ in $\mathbf{R}^3$ such that $U - Q$ is homeomorphic with $\partial U \times (0, 1]$.

As every wild arc in $\mathbf{R}^3$ is tame in $\mathbf{R}^4$, there exists a continuous map $\tau : \mathbf{R}^4 \to \mathbf{R}^4$ with $\tau(Q) = \mathbf{0}$ and $\tau | \mathbf{R}^4 - Q$ is a diffeomorphism onto $\mathbf{R}^4 - \{\mathbf{0}\}$.

Another such $Q \subset \mathbf{R}^3$ (with $\mathbf{R}^3 - Q$ simply connected) can be constructed by using the Whitehead example ([5] and [6] Chapter 7).

## § 4. *The construction of* $Q \subset \mathbf{R}^n \subset \mathbf{R}^{n+1}$ *for* $n \geqslant 4$.

Newman [4] and Mazur [3] proved that for every $n \geqslant 4$ there exist compact contractible manifolds $X^n$ with:

(a) $\pi_1(\partial X^n) \neq 0$,
(b) $X^n$ can be embedded in $\mathbf{R}^n$,
(c) $X^n \times [0, 1]$ is, up to edges in the boundary, diffeomorphic with

$$D^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \,\Big|\, \sum_{i=1}^{n+1} x_i^2 \leqslant 1\}.$$

(c) implies that the double of $X^n$ is diffeomorphic with $S^n$.

Let $M^n$ be such a manifold and let $G$ be the fundamental group of its boundary.

Definition 3. Let $K$ and $L$ be $n$ dimensional manifolds with boundary. $K \boxplus_f L$ the *connected sum* of $K$ and $L$ *along the boundary* is the $n$-manifold formed by an identification $f$ of $D^{n-1} \subset \partial K$ with $D^{n-1} \subset \partial L$ and by smoothing the boundary. We omit $f$ from the notation. It is easy to see that $\partial(K \boxplus L) = (\partial K) \# (\partial L)$ is the usual connected sum of $\partial K$ and $\partial L$. We now construct in $\mathbf{R}^n$ a sequence $\{Q_i\}_{i=0}^{\infty}$ of $n$-manifolds with boundary having the following properties:

1. $Q_0 = D^n$,
2. $Q_{i+1} \subset \operatorname{int} Q_i$,
3. $Q_{i+1}$ is diffeomorphic with $Q_i \boxplus M^n$,
4. $Q_i - \operatorname{int} Q_{i+1}$ is diffeomorphic with $(\partial Q \times [0, 1]) \boxplus M^n$.

$Q$ is defined to be $\bigcap_{i=0}^{\infty} Q_i$. We construct $\{Q_i\}_{i=0}^{\infty}$ by induction. Suppose we have already $\{Q_i\}_{i=0}^{k}$ for some $k \geqslant 0$, we proceed to construct $Q_{k+1}$.

As $Q_k$ is a manifold (with smooth boundary) we can choose a collar of $\partial Q_k$ (i.e. we choose an embedding

$$\Psi : (\partial Q_k \times [0, 1], \partial Q_k \times \{0\}) \to (Q_k, \partial Q_k)).$$

Let $\Phi : D^n \to \operatorname{int} Q_k$ be an embedding with

$$\Phi(\{(x_1, \dots, x_n) \mid \sum_{i=1}^{n} x_i^2 \leqslant 1 \text{ and } x_1 \geqslant 0\}) = \Phi(D^n) \cap Im(\Psi).$$

As the double of $M^n$ is $S^n$, one can devide $D^n$ in two parts $A_1$ and $A_2$ by a properly embedded submanifold $V^{n-1}$ (i.e. $V^{n-1} = A_1 \cap A_2$ and $D^n = A_1 \cup A_2$), such that $A_1$ and $A_2$ both are diffeomorphic with $M^n$ (up to edges in the boundary) and $A_1 \cap \partial D^n$, $A_2 \cap \partial D^n$ are diffeomorphic with $D^{n-1}$.

One can choose $V^{n-1} \subset D^n$ so that

$$\{(x_1, \dots, x_n) \mid \tfrac{1}{2} \leqslant \sum_{i=1}^{n} x_i^2 \leqslant 1 \text{ and } x_1 = 0\} \subset V^{n-1}$$

and

$$A_1 \cap \partial D^n = \{(x_1, \dots, x_n) \mid \sum_{i=1}^{n} x_i^2 = 1 \text{ and } x_1 \geqslant 0\}.$$

Now $Q_{k+1}$ is defined by:

$$Q_k - \operatorname{int} Q_{k+1} = (\Psi(\partial Q_k \times [0, 1]) - \Phi(D^n)) \cup \Phi(A_1);$$

see figure 1.

$Q_{k+1}$ satisfies all the required properties. We now conclude the proof of the theorem with the following proposition:

Proposition 2: The triple $(Q = \bigcap_{i=0}^{\infty} Q_i, \mathbf{R}^n, \mathbf{R}^{n+1})$ has the properties (i) and (ii) announced in proposition 1.

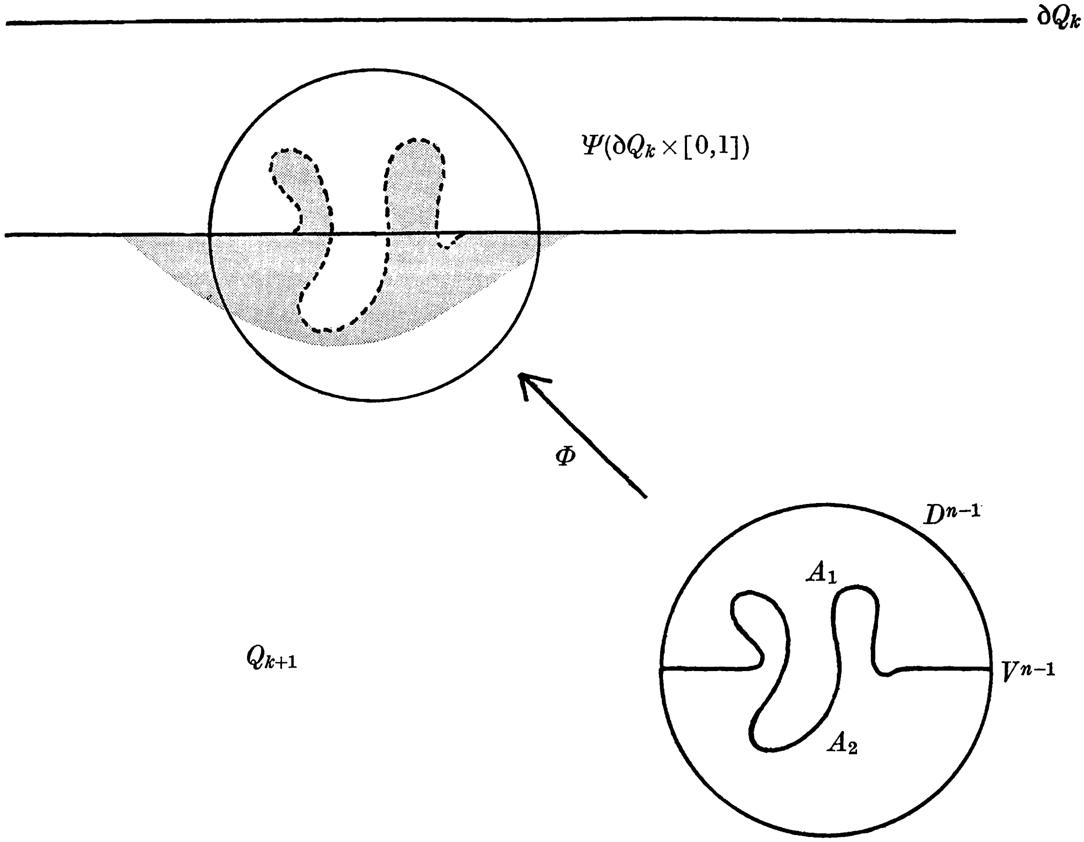


Fig. 1

Proof: We first list some properties of $\{Q_i\}_{i=0}^{\infty}$:

($\alpha$) $\partial Q_i$ is diffeomorphic with $(\partial Q_{i-1}) \# (\partial M^n)$ which again is diffeomorphic with $(\partial M^n)^{\# i} = \partial M^n \# \partial M^n \# \dots \# \partial M^n$ ($i$ times),

($\beta$) $\pi_1(\partial Q_i) = G^{*i} = G \star G \star \dots \star G$ ($i$ times),

($\gamma$) there exists a homotopy equivalence $h : \partial Q_i \to (Q_i - \operatorname{int} Q_{i+k})$ for all $k, i \geqslant 0$.

From $\gamma$ it follows that the inclusion $\partial Q_i \to (Q_i - Q)$ is a weak homotopy equivalence for all $i$; from this and $\beta$ it follows that the map $\pi_1(Q_{i+k} - Q) \to \to \pi_1(Q_i - Q)$, induced by the inclusion, is always surjective but never injective for all $i \geqslant 0$ and $k \geqslant 1$ (the base point is some point in $Q_{i+k} - Q_k$).

We will now prove (ii). Suppose there exists a closed neighbourhood $U$ of $Q$ in $\mathbf{R}^n$ such that $U - Q$ is homeomorphic with $\partial U \times (0, 1]$. Then there is a basis of closed neighbourhoods $\{U_t\}_{t \in (0,1]}$ of $Q$ in $\mathbf{R}^n$ such that $U - U_t$ is homeomorphic with $\partial U \times (t, 1]$.

The sequence $\{Q_i\}_{i=0}^{\infty}$ also forms a basis of closed neighbourhoods of $Q$ in $\mathbf{R}^n$. We can choose $0 < i < j < k$ and $t, t' \in (0, 1]$ with $Q_k \subset U_{t'} \subset Q_j \subset \subset U_t \subset Q_i \subset U_1 = U$. Consider the following diagram (the base point is somewhere in $Q_k - Q$):

$$\begin{tikzcd}
\pi_1(Q_i - Q) \arrow[r, "\alpha_1"] & \pi_1(U_1 - Q) \\
\pi_1(Q_j - Q) \arrow[u, "\alpha_4"] \arrow[r, "\alpha_2"'] & \pi_1(U_t - Q) \arrow[u, "\alpha_6"'] \arrow[ul, "\alpha_5"'] \\
\pi_1(Q_k - Q) \arrow[u, "\alpha_7"] \arrow[r, "\alpha_3"'] & \pi_1(U_{t'} - Q) \arrow[u, "\alpha_9"'] \arrow[ul, "\alpha_8"']
\end{tikzcd}$$

All maps are induced by inclusions, so the diagram commutes.

From the definition of $U_t$ it follows that $\alpha_6$ and $\alpha_9$ are isomorphisms, so $\alpha_5$ and $\alpha_8$ are injective. We proved that $\alpha_4$ and $\alpha_7$ are surjective, so $\alpha_5$ and $\alpha_8$ must be surjective, so $\alpha_5$, $\alpha_8$ and $\alpha_9$ are isomorphisms; this implies that $\alpha_4$ is an isomorphism. We proved however that $\alpha_4$ cannot be injective so we have a contradiction which proves (ii).

To prove (i) it is sufficient to show that there exists a basis $\{W_i\}_{i=0}^{\infty}$ of closed neighbourhoods of $Q$ in $\mathbf{R}^{n+1}$ such that each $W_i$ is diffeomorphic with $D^{n+1}$ and $W_i \supset W_{i+1}$.

In order to construct $W_i$ we first consider

$$\tilde{W}_i = \{(x_1, \ldots, x_{n+1}) \mid (x_1, \ldots, x_n) \in Q_i \text{ and } |x_{n+1}| \leqslant i^{-1}\} \subset \mathbf{R}^{n+1}.$$

From the construction of $Q_i$ and the fact that $M^n \times [0, 1] \cong D^{n+1}$, it follows that $\tilde{W}_i$ is, up to edges in the boundary, diffeomorphic with $D^{n+1}$. We obtain $W_i$, very close to $\tilde{W}_i$, by suitably smoothing the edges in $\partial(\tilde{W}_i)$. Then $\{W_i\}_{i=0}^{\infty}$ has the required properties.

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